A local search heuristic for a mixed integer nonlinear integrated airline schedule planning problem

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Abstract

In this paper we present a local search heuristic method for an integrated airline scheduling, fleeting and pricing model. The integrated model simultaneously optimizes the decisions of schedule design, fleet assignment, seat allocation, pricing and considers passengers' spill and recapture. The resulting problem is a mixed integer non-convex problem due to the explicit representation of a demand model which guides the revenue management decisions. The local search heuristic tackles the complexity of the problem decomposing the problem into two simplified versions of the integrated model. The first model is a fleet assignment model where the pricing decision is fixed. The fleet assignment sub-model is a mixed integer linear problem. The second model is a revenue management model where the fleet assignment decision, i.e., the transportation capacity, is fixed. This revenue sub-model is a continuous nonlinear problem. These sub-models are solved in an iterative way with intelligent local search mechanisms. Price sampling is used for a local search on price and variable neighborhood search techniques are used for exploring superior fleet assignment decisions. Metaheuristic mechanisms permit to escape from local optima. The local search heuristic is presented in comparison to two other heuristic approaches: a heuristic procedure provided by an open-source generic MINLP solver and a sequential approach which mimic the current practice of airlines. The three approaches are tested on a set of experiments with different problem sizes. The local search heuristic outperforms the two other approaches in terms of the quality of the solution and computational time.

Keywords: integrated schedule planning; mixed integer nonlinear problem; non-convexity; heuristics; local search; variable neighborhood search

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1 Introduction

The design of a competitive schedule and the decisions related to the fleet assignment are critical for airline's profitability. Schedule planning decisions are taken in large advance with respect to the day of operations according to an estimate of the transportation demand. Once the decisions on scheduling and fleeting are published, few changes can be made to take into account demand fluctuation. Furthermore, the demand is given at the itinerary level, however the capacity must be decided at the flight level. We refer to Sherali et al. (2006) for a review on airline fleet assignment literature. The itinerary-based fleet assignment model (IFAM, Barnhart et al., 2002) is well accepted in the literature in order to be able to better represent the network effects of scheduling and fleeting decisions. With a further attempt to better handle the network effects, demand correction terms and recapture effects are included in fleet assignment models (Lohatepanont and Barnhart, 2004). In case of capacity shortage, a portion of passengers that cannot be accommodated on the desired flight may accept to be redirected to other itineraries in the same market. When appropriately modeled, the proportion of recaptured passengers, called recapture ratio, provides flexibility for airlines in their capacity planning.

In the literature, there are studies dedicated to the integration of network effects in fleet assignment models. Anyway, in such models the estimation on the revenue usually fails to represent reality: either the demand and price are assumed to be given as an input or simple revenue models are included. We refer the reader to Talluri and van Ryzin (2004a) for a review on revenue management models. The need for the incorporation of more realistic revenue functions is underlined by Barnhart et al. (2009). Authors make assumptions on the revenue functions that allow for the design of solution methodologies for the fleet assignment problem. One fundamental assumption is that the revenue is not a function of the price.

Talluri and van Ryzin (2004b) integrate discrete choice modeling into the single-leg, multiple-fare-class revenue management model that determines the subset of fare products to offer at each point in time. Authors provide the characterization of optimal policies under a general choice model of demand. Schön (2008) shows the integration of different choice models into an integrated schedule design and fleet assignment model. The studied demand models include logit and nested logit formulations with simple structures where the only explanatory variable is the price of the itineraries. The inverse demand function is used instead of the explicit logit formula in order to obtain a convex formulation. The convex model is then solved using a benders decomposition approach. However, in case of having more than one variable or in the presence of disaggregate variables the inverse operation cannot be used and therefore convexity cannot be guaranteed.

Recently, Atasoy et al. (2013) introduce an integrated airline scheduling, fleeting and pricing model formulated as a non-convex mixed integer nonlinear problem (MINLP). The pricing is integrated through an itinerary choice model which shares some similarities with Schön (2008). One distinction of the considered model is the integration of a choice model which is estimated on real data. The dataset is a mixed revealed preferences and stated preferences data which enables to enrich the model with explanatory variables additional to price. Another important contribution is that the choice model is explicitly included in the optimization model rather than an inverse demand function, as proposed by Schön (2008). This gives flexibility for the further extensions of the demand model with more explanatory variables and/or disaggregate level data. The itinerary choice model is specific to economy and business classes and enables to optimize the decisions on capacity allocation for each class. Furthermore, the choice model is adapted to appropriately consider the spill and recapture effects. The resulting model simultaneously optimizes the schedule planning, fleet assignment, pricing and capacity allocation to classes.

Atasoy et al. (2013) report the added value of the integrated scheduling, fleeting and pricing model by solving the monolithic model with an open-source solver. However, the solver is designed for convex problems, which is not the case of the integrated model. In this paper we present a local search heuristic based on two sub-models of the problem. Inspired by the idea of D'Ambrosio et al. (2012), we fix either the fleet or the revenue part of the integrated model in order to obtain simplified models. When we fix the pricing part, we obtain a mixed integer linear problem (MILP). This sub-model is a fleet assignment problem where the price and the recapture ratios are inputs. When we fix the fleet assignment decisions, we obtain a non-convex nonlinear problem (NLP). Therefore it consists of a revenue management model with a given capacity. The two sub-models are solved in an iterative procedure where local search techniques are used to explore alternative feasible solutions. Local search techniques include price sampling that is used to visit new fleet assignment solutions with different price inputs. Furthermore a variable neighborhood search (Hansen and Mladenović, 2001) is developed so that a sub-set of the fleet assignments are fixed and kept for the next iteration based on the quality of the incumbent solution. We also design a tabu search mechanism (Glover, 1990) to prevent multiple visits to the same solution for a number of consequent iterations. The main contribution of the paper is a local search heuristic which is designed to handle the difficulties of the model thanks to a combination of the above-mentioned techniques. The interactions between supply and demand models are exploited and as a result, this combination provides better quality feasible solutions for realistic size instances compared to other two heuristic approaches: an MINLP solver (BONMIN, Bonami et al., 2008) and a sequential approach which mimics the current practice of airlines. The presented local search heuristic can easily be used by practitioners for the solution of integrated scheduling, fleeting and pricing decisions.

The rest of the paper is organized as follows. In section 2 we present the integrated airline scheduling, fleeting and pricing model that is introduced by Atasoy et al. (2013). In section 3 we introduce the three heuristic approaches for the integrated model. Section 4 describes the data instances used for the experiments throughout the analysis. In section 5 we provide experimental results on the performance of the three approaches. We analyze the results in terms of the quality of the solution and computational time. Finally we conclude the paper and provide future directions in section 6.

2 Integrated airline scheduling, fleeting and pricing model

For the sake of self completeness, we briefly report the integrated airline scheduling, fleeting and pricing model introduced by Atasoy et al. (2013). The model is based on a time-space network of an airline's schedule. Every node in the network represents a departure or arrival event at an airport at a specific time. Every arc in the network models a feasible connection between events, either a flight arc between two different airports or a ground arc at the same airport. As the model includes scheduling decisions, a set of optional flights is considered. These flights represent the set of potential changes with respect to a baseline schedule. The parameters of the model are reported in Table 1 and the decision variables are presented in Table 2.

Table 1: Parameters of the integrated model

Set	Definition
F	set of flight legs indexed by f
F_M	set of mandatory flight legs
F_O	set of optional flight legs
CT	set of flights flying at count time
A	set of airports indexed by a
K	set of fleet types indexed by k
T	set of time of the events in the network indexed by t
$N(k,a,t)$	set of the nodes in the time-space network for fleet type k, airport a and time t
$\text{In}(k,a,t)$	set of inbound flight legs for node (k,a,t)
$\text{Out}(k,a,t)$	set of outbound flight legs for node (k,a,t)
H	set of cabin classes indexed by h
S^h	set of market segments indexed by s, for cabin class h
I_s	set of itineraries in segment s, indexed by i
$I_s^{'}$	set of no-revenue itineraries, $I_s^{'} \in I_s$
Parameter	**Definition**
$C_{k,f}$	operating cost for flight f when operated by fleet type k
R_k	available number of planes for type k
Q_k	the capacity of fleet type k in number of seats
$min\text{E}_a^-$	the time just before the first event at airport a
$max\text{E}_a^+$	the time just after the last event at airport a
$\delta_{i,f}$	1 if itinerary i uses flight leg f, 0 otherwise
D_s	the total unconstrained demand for segment s
LB_i	the lower bound on the price of the itinerary i
UB_i	the upper bound on the price of the itinerary i
z_i	the vector of explanatory variables for itinerary i
β	the vector of parameters of the logit model
V_i	the utility of itinerary i

Table 2: Decision variables of the integrated model

Variable	Definition
	Schedule planning
$x_{k,f}$	binary variable, 1 if fleet type k is assigned to flight f, 0 otherwise
y_{k,a,t^-}	continuous variable, the number of type k planes at airport a just before time t
y_{k,a,t^+}	continuous variable, the number of type k planes at airport a just after time t
	Revenue management: all variables are continuous
d_i	demand of itinerary i
$\tilde{d}_i$	demand share of itinerary i based on the logit model, which serves as an upper bound for d_i
p_i	price of itinerary i (fixed for no-revenue itineraries)
$t_{i,j}$	redirected passengers from itinerary i to itinerary j
$b_{i,j}$	recapture ratio for the passengers spilled from itinerary i and redirected to itinerary j
$\pi_{k,f}^h$	assigned seats for flight f in a type k plane for cabin class h

$$
\begin{aligned}
\max \sum_{h\in H}\sum_{s\in S^h}\sum_{i\in(I_s\setminus I_s^{'})}&(d_i-\sum_{j\in I_s}t_{i,j}+\sum_{j\in(I_s\setminus I_s^{'})}t_{j,i}b_{j,i})p_i \\
&-\sum_{\substack{k\in K\\ f\in F}}C_{k,f}x_{k,f} && && (1)\\
\text{s.t.} \sum_{k\in K}x_{k,f}&=1 && \forall f\in F^M && (2)\\
\sum_{k\in K}x_{k,f}&\leq 1 && \forall f\in F^O && (3)\\
y_{k,a,t^-}+\sum_{f\in \mathrm{In}(k,a,t)}x_{k,f}&=y_{k,a,t^+}+\sum_{f\in \mathrm{Out}(k,a,t)}x_{k,f} && \forall [k,a,t]\in N && (4)\\
\sum_{a\in A}y_{k,a,\min\mathrm{E}_a^-}+\sum_{f\in CT}x_{k,f}&\leq R_k && \forall k\in K && (5)\\
y_{k,a,\min\mathrm{E}_a^-}&=y_{k,a,\max\mathrm{E}_a^+} && \forall k\in K, a\in A && (6)\\
\sum_{s\in S^h}\sum_{i\in(I_s\setminus I_s^{'})}\delta_{i,f}(d_i-\sum_{j\in I_s}t_{i,j}&+\sum_{j\in(I_s\setminus I_s^{'})}t_{j,i}b_{j,i})\\
&\leq \sum_{k\in K}\pi_{k,f}^h && \forall h\in H, f\in F && (7)\\
\sum_{h\in H}\pi_{k,f}^h&\leq Q_k x_{k,f} && \forall f\in F, k\in K && (8)\\
\sum_{j\in I_s}t_{i,j}&\leq d_i && \forall h\in H, s\in S^h, i\in(I_s\setminus I_s^{'}) && (9)\\
\tilde{d}_i&=D_s\frac{\exp\left(V_i(p_i,z_i;\beta)\right)}{\sum_{j\in I_s}\exp\left(V_j(p_j,z_j;\beta)\right)} && \forall h\in H, s\in S^h, i\in I_s && (10)\\
b_{i,j}&=\frac{\exp\left(V_j(p_j,z_j;\beta)\right)}{\sum_{k\in I_s\setminus\{i\}}\exp\left(V_k(p_k,z_k;\beta)\right)} && \forall h\in H, s\in S^h, i\in(I_s\setminus I_s^{'}), j\in I_s && (11)\\
x_{k,f}&\in\{0,1\} && \forall k\in K, f\in F && (12)\\
y_{k,a,t}&\geq 0 && \forall [k,a,t]\in N && (13)\\
\pi_{k,f}^h&\geq 0 && \forall h\in H, k\in K, f\in F && (14)\\
0\leq d_i&\leq \tilde{d}_i && \forall h\in H, s\in S^h, i\in I_s && (15)\\
LB_i\leq p_i&\leq UB_i && \forall h\in H, s\in S^h, i\in(I_s\setminus I_s^{'}) && (16)\\
t_{i,j}&\geq 0 && \forall h\in H, s\in S^h, i\in(I_s\setminus I_s^{'}), j\in I_s && (17)\\
b_{i,j}&\geq 0 && \forall h\in H, s\in S^h, i\in(I_s\setminus I_s^{'}), j\in I_s && (18)
\end{aligned}
$$

Objective function(1) maximizes the profit calculated by the revenue minus operating costs. Constraints (2)-(6) are specific for the fleet assignment process. Constraints (2) ensure that the mandatory flights are operated. Constraints (3) are for the optional flights that can be canceled. Constraints (4) maintain the flow conservation of the fleet at every node of the state-space network. Constraints (5) limit the use of each aircraft type according to the fleet size. It is assumed that the network configuration at the beginning and at the end of the period, which is one day, is the same in terms of the number of aircraft at each airport (6).

The relation between the supply capacity and the actual demand is maintained by constraints (7) which ensure that the assigned capacity for a flight satisfies the demand for the corresponding itineraries. The actual demand is composed of the potential demand of the itinerary minus the spilled passengers plus the recaptured passengers from other itineraries. The same constraints ensure that the itineraries do not realize any demand if any of the corresponding flight legs is canceled. We let the allocation of business and economy seats to be decided by the model as a revenue management decision. Therefore, constraints (8) ensure that the total allocated seats does not exceed the capacity of the aircraft.

Demand related constraints include constraints (9) which ensure that the total redirected passengers from itinerary i to all other itineraries, including the no-revenue options, do not exceed its realized demand. The demand given by the logit model, $\tilde{d}_i$, is provided as in the constraints (10). This formula gives the demand for each itinerary in a market segment depending on the utilities of all the available itineraries in the same segment. The utility of each itinerary, V_i, depends on the price, p_i and a vector of explanatory variables, z_i, consisting of trip length, departure time of day and the number of stops. In the present model, the price is the only policy variable which can be directly controlled. The other explanatory variables have an indirect effect on the scheduling decisions. The β parameters are estimated based on a mixed revealed preferences and stated preferences data. The details on the logit model, the estimation procedure and the estimation results are provided in Atasoy and Bierlaire (2012).

The recapture ratio is modeled with a similar logit formulation as seen in the constraints (11). When a passenger is redirected from itinerary i to itinerary j, the probability of passenger's acceptance is given by the market share of itinerary j among the available itineraries in the market segment excluding the itinerary i. The spill phenomenon is assumed to be the decision of the airline rather than a market equilibrium. Market reacts only with the recapture ratios. Finally, restriction on variables' domain are ensured by constraints (12)-(18).

The resulting model is a non-convex MINLP. The non-convexity comes from the integration of the logit model.

The added value of the integrated model is presented by Atasoy et al. (2013) compared to state-of-the-art models. The integrated model is found to take superior scheduling and fleeting decisions due to the explicit supply-demand interactions. When there is a potential in increasing the price of some itineraries, the integrated model decides to do so with a less capacity and increases the profit. Similarly, when there is a room for attracting more passengers with a small decrease in the price, the integrated model increases the capacity and obtains a higher profit.

Atasoy et al. (2013) perform the mentioned tests over the integrated model using the BONMIN solver. As discussed in section 5, BONMIN is computationally inefficient for solving the full integrated model and cannot provide good quality feasible solutions for medium size instances even in 12 hours. These limitations necessitate the development of a more efficient method which is the main motivation of this paper.

3 Heuristic approaches

In this section we present three heuristic approaches for the solution of the integrated airline schedule planning model. The first two approaches serve as references for the local search heuristic.

3.1 BONMIN solver for the integrated model

BONMIN[1] is an open-source solver proposed by Bonami et al. (2008) and designed to solve convex MINLPs. As it is designed to be an exact method for convex problems, it can be only considered as a heuristic for solving the integrated model. The main methods embedded in the solver are branch and bound and polyhedral outer approximation.

3.2 Sequential approach

As a second heuristic approach for the solution of the integrated model, we mimic the current practice of airlines where revenue management decisions are taken with a fixed capacity provided by the schedule planning process. Talluri and van Ryzin (2004a) provides the state-of-the-art revenue management models and states the fact that most revenue management models assume the capacity is given and fixed. A similar sequential approach is utilized by Lohatepanont (2002) in the context of a sensitivity analysis for an itinerary-based fleet assignment model.

We represent the sequential approach with two sub-models of the integrated model. The first sub-model is the fleet assignment model (FAM), where the price of the itineraries are inputs and the remaining decisions are optimized with the given price and demand. The optimized decisions are the schedule design, fleet assignment, seat allocation and the number of spilled passengers. This model is indeed an extended version of the state-of-the-art fleet assignment models (Lohatepanont and Barnhart, 2004) with more advanced methodology on the spill and recapture effects. Since the pricing decision is excluded, the prices of the itineraries (p) are fixed. The demand given by the logit ($\tilde{d}$) and the recapture ratios (b) are also parameters that are calculated with the given price. Therefore we represent them by P, $\tilde{D}$, and B for clarification purposes. The FAM is a MILP and given as follows:

[1] https://projects.coin-or.org/Bonmin

$$z^*_{\mathbf{FAM}} = \max \sum_{h\in H} \sum_{s\in S^h} \sum_{i\in(I_s \setminus I'_s)} (d_i - \sum_{j\in I_s} t_{i,j} + \sum_{j\in(I_s\setminus I'_s)} t_{j,i}B_{j,i})P_i \tag{19}$$

$$- \sum_{\substack{k\in K\\ f\in F}} C_{k,f}x_{k,f} \tag{20}$$

$$\text{s.t.} \sum_{k\in K} x_{k,f} = 1 \qquad \forall f\in F^M \tag{21}$$

$$\sum_{k\in K} x_{k,f} \leq 1 \qquad \forall f\in F^O \tag{22}$$

$$y_{k,a,t^-} + \sum_{f\in \mathrm{In}(k,a,t)} x_{k,f} = y_{k,a,t^+} + \sum_{f\in \mathrm{Out}(k,a,t)} x_{k,f} \qquad \forall [k,a,t]\in N \tag{23}$$

$$\sum_{a\in A} y_{k,a,\mathrm{minE}_a^-} + \sum_{f\in CT} x_{k,f} \leq R_k \qquad \forall k\in K \tag{24}$$

$$y_{k,a,\mathrm{minE}_a^-} = y_{k,a,\mathrm{maxE}_a^+} \qquad \forall k\in K, a\in A \tag{25}$$

$$\sum_{s\in S^h} \sum_{i\in(I_s\setminus I'_s)} \delta_{i,f}(d_i - \sum_{j\in I_s} t_{i,j} + \sum_{j\in(I_s\setminus I'_s)} t_{j,i}B_{j,i})$$

$$\leq \sum_{k\in K} \pi^h_{k,f} \qquad \forall h\in H, f\in F \tag{26}$$

$$\sum_{h\in H} \pi^h_{k,f} \leq Q_k x_{k,f} \qquad \forall f\in F, k\in K \tag{27}$$

$$\sum_{j\in I_s} t_{i,j} \leq d_i \qquad \forall h\in H, s\in S^h, i\in (I_s\setminus I'_s) \tag{28}$$

$$x_{k,f}\in\{0,1\} \qquad \forall k\in K, f\in F \tag{29}$$

$$y_{k,a,t}\geq 0 \qquad \forall [k,a,t]\in N \tag{30}$$

$$\pi^h_{k,f}\geq 0 \qquad \forall h\in H, k\in K, f\in F \tag{31}$$

$$0\leq d_i\leq \tilde{D}_i \qquad \forall h\in H, s\in S^h, i\in I_s \tag{32}$$

$$t_{i,j}\geq 0 \qquad \forall h\in H, s\in S^h, i\in(I_s\setminus I'_s), j\in I_s \tag{33}$$

The second sub-model is a revenue management model (RMM) with a fixed capacity. The available seat capacity for every flight is given as input. This model is a non-convex NLP. Since the fleet assignment decisions of x and y are fixed they are parameters for the RMM and represented by X and Y for the ease of explanation. The RMM is provided as follows:

$$z^*_{\mathbf{RMM}} = \max \sum_{h\in H} \sum_{s\in S^h} \sum_{i\in(I_s\setminus I_s^{'})} (d_i - \sum_{j\in I_s} t_{i,j} + \sum_{j\in(I_s\setminus I_s^{'})} t_{j,i}b_{j,i})p_i \quad (34)$$

$$\text{s.t.} \sum_{s\in S^h} \sum_{i\in(I_s\setminus I_s^{'})} \delta_{i,f}(d_i - \sum_{j\in I_s} t_{i,j} + \sum_{j\in(I_s\setminus I_s^{'})} t_{j,i}b_{j,i})$$

$$\leq \sum_{k\in K} \pi^h_{k,f} \qquad \forall h \in H, f \in F \quad (35)$$

$$\sum_{h\in H} \pi^h_{k,f} \leq Q_k X_{k,f} \qquad \forall f \in F, k \in K \quad (36)$$

$$\sum_{j\in I_s} t_{i,j} \leq d_i \qquad \forall h \in H, s \in S^h, i \in (I_s \setminus I_s^{'}) \quad (37)$$

$$\tilde{d}_i = D_s \frac{\exp\left(V_i(p_i, z_i; \beta)\right)}{\sum_{j\in I_s} \exp\left(V_j(p_j, z_j; \beta)\right)} \qquad \forall h \in H, s \in S^h, i \in I_s \quad (38)$$

$$b_{i,j} = \frac{\exp\left(V_j(p_j, z_j; \beta)\right)}{\sum_{k\in I_s\setminus\{i\}} \exp\left(V_k(p_k, z_k; \beta)\right)} \qquad \forall h \in H, s \in S^h, i \in (I_s \setminus I_s^{'}), j \in I_s \quad (39)$$

$$\pi^h_{k,f} \geq 0 \qquad \forall h \in H, k \in K, f \in F \quad (40)$$

$$0 \leq d_i \leq \tilde{d}_i \qquad \forall h \in H, s \in S^h, i \in I_s \quad (41)$$

$$LB_i \leq p_i \leq UB_i \qquad \forall h \in H, s \in S^h, i \in (I_s \setminus I_s^{'}) \quad (42)$$

$$t_{i,j} \geq 0 \qquad \forall h \in H, s \in S^h, i \in (I_s \setminus I_s^{'}), j \in I_s \quad (43)$$

$$b_{i,j} \geq 0 \qquad \forall h \in H, s \in S^h, i \in (I_s \setminus I_s^{'}), j \in I_s \quad (44)$$

The sequential approach first solves the FAM with the average price values provided in the dataset. It optimizes the schedule design and fleet assignment ($x_{k,f}$, $y_{k,a,t}$). These decisions on the capacity are given as inputs to the next step which is the solution of the RMM. It provides the price of each itinerary (p_i), the actual demand (d_i), the allocated seats to each class ($\pi^h_{k,f}$) and the number of spilled passengers ($t_{i,j}$).

3.3 Local search heuristic

The third heuristic approach is the main contribution of this paper. It is based on the sequential approach and the use of appropriate local search mechanisms. The main shortcoming of the sequential approach is that the capacity provided by the FAM cannot make use of the information on the revenue since it runs with fixed price and demand for the itineraries. FAM is not able to account for the potential in changing the pricing decisions in order to shape the demand and come up with more profitable schedule planning. Therefore a local search heuristic is developed answering to this lack of interaction between planning and revenue decisions. The neighborhood is defined by local search techniques which provide alternative schedule planning decisions. Namely, the alternative solutions for the $x_{k,f}$ variables constitute neighborhood solutions.

The first local search mechanism is *price sampling* which reveals the potential improvement on the revenue as a consequence of the adjustments of the price. The second mechanism is *variable neighborhood search* which keeps a varying subset of fleet assignment solutions fixed in the model based on the quality of the solution. Both of the procedures are based on the number of spilled passengers. This information is found to be important since the spilled passengers are potential revenue sources. The local search procedures are then capable of realizing the impact of planning decisions on the revenue and directing the algorithm towards good feasible solutions.

3.3.1 Price sampling

As mentioned previously, the FAM considers fixed price and fixed demand values. In order to visit alternative solutions, the model is iteratively solved drawing different price samples and different itinerary demands. The sampling procedure takes into account the rate of spilled passengers resulting from the solution of the RMM in the previous iteration. The spill rate of a flight is defined as the average spilled passengers divided by the total demand for the flight (McGill, 1989; Belobaba, 2006). Similarly, for every itinerary i, the SR_i^g rate is defined as the number of spilled passengers over the realized demand in iteration g as follows:

$$SR_i^g = \frac{\sum\limits_{j \in I_s} t_{i,j}^g}{d_i^g} \qquad \forall h \in H, s \in S^h, i \in (I_s \setminus I_s^{'})$$

In price sampling, according to the solution of RMM, the price of an itinerary in iteration g is altered based on the number of spilled passengers in the previous iteration $g-1$. The price is increased if that itinerary presents a lower SR_i^{g-1} rate compared to the average rate, which is denoted by SR_{mean}^{g-1}. A random price value is uniformly drawn between the current price value and the upper bound. On the other hand, the price is decreased if the spill rate is higher than SR_{mean}^{g-1}. A random price value is uniformly drawn between the lower bound and the current price value. This price sampling is given as follows:

$$p_i^g = \begin{cases} \text{unirand}(p_i^{g-1}, UB_i) & \text{if } SR_i^{g-1} \leq SR_{mean}^{g-1} \\ \text{unirand}(LB_i, p_i^{g-1}) & \text{otherwise} \end{cases} \qquad \forall h \in H, s \in S^h, i \in (I_s \setminus I_s^{'})$$

3.3.2 Variable neighborhood search - VNS

While neighborhood schedule planning solutions are being explored, a subset of fleet assignments is fixed, i.e. some flights are kept assigned to the same aircraft, for a number of iterations in order to keep the good fleet assignment solutions. The number of fixed assignments is represented by n_{fixed}. The variable neighborhood mechanism is embedded in such a way that n_{fixed} is altered according to the quality of the solution. If a better solution is obtained, n_{fixed} is increased in the next iteration which is referred as *intensification*. On the other hand, when there is no improvement for a subsequent number of iterations, a *diversification* is applied, i.e. n_{fixed} is decreased in order to better explore the feasible region.

The set of fixed assignments is represented by Ł which has n_{fixed} elements. Each fixed assignment ℓ indicates a fleet type k_ℓ^{fixed} and a flight f_ℓ^{fixed}. This fixing is maintained by the constraint given by equation (45). Therefore the FAM considered for the local search heuristic is represented by (20)-(33) and (45).

$$x_{k_\ell^{fixed}, f_\ell^{fixed}} = 1 \qquad \forall \ell \in Ł \tag{45}$$

The decision to fix a fleet assignment is taken considering the number of spilled passengers. In other words, an aircraft type is assigned to the corresponding flight in the current solution with a probability which depends on the number of passengers spilled from itineraries involving that flight. Intuitively the smaller the spill from a flight, the higher the probability that the flight-aircraft pair is fixed in the current iteration. The set of flights which are flown at iteration g is represented by F^g_{flown}. The spill rate of a flight, SR^g_f, is the sum of the spill rates of all itineraries involving flight f as stated in equation (46).

$$SR^g_f = \sum_{h \in H} \sum_{s \in S^h} \sum_{i \in (I_s \setminus I'_s)} \delta_{i,f} SR^g_i \qquad \forall f \in F^g_{flown} \tag{46}$$

The maximum SR^g_f rate among all the flights in F^g_{flown} is denoted by SR^g_{max}. The probability of fixing the assignment of flight f at iteration g, prob^g_f, is obtained according to the number of spilled passengers at iteration $g-1$ as provided in equation (47). It is proportional to the gap between the maximum spill rate and the spill rate of flight f. Therefore, the probability is higher when the number of spilled passengers is lower.

$$\text{prob}^g_f = \frac{SR^{g-1}_{max} - SR^{g-1}_f}{\sum_{j \in F^{g-1}_{flown}} (SR^{g-1}_{max} - SR^{g-1}_j)} \qquad \forall f \in F^{g-1}_{flown} \tag{47}$$

3.3.3 Tabu search

The local search mechanisms allow to visit alternative solutions. In order to prevent visiting the same solutions and therefore to fasten the process a tabu search framework is considered. The explored fleet assignment solutions ($x_{k,f}$) are kept in a tabu list. The size of the tabu list is determined according to the size of the instance studied. For small instances with a small number of flights and aircraft types the tabu list keeps less fleet assignment solutions compared to larger instances. When a new set of solutions is introduced in the list, the last one is removed automatically if the maximum size is reached.

3.3.4 The complete local search heuristic

The local search heuristic consists of iterations each of which solves FAM and RMM subsequently. As mentioned previously, FAM is solved by fixing the revenue part and RMM is solved by fixing the schedule planning decisions. This fixing is embedded in an iterative process similar to the idea of D'Ambrosio et al. (2012). The iterative process is carried out with the local search mechanisms defined in sections 3.3.1 and 3.3.2. These local search techniques enable to visit good quality neighborhood solutions.

The procedure is presented by Algorithm 1. The iteration continue until the time limit, time_{max}. The decision variables of the model are represented by the same notation in the algorithm. The price variables are initialized with the given price values in the data set. This implies that the first iteration of the local search heuristic is actually the sequential approach. However with the local search mechanisms this sequential approach solution is improved.

$n_{\min}$ and $n_{\max}$ are defined as the minimum and maximum number of fixed assignments according to the data instance. *notImpr* is the number of subsequent iterations where there was no improvement in the best objective function value, z^*. *tabuList* is the tabu list of size *tabuListSize*, that consists of the fleet assignment solutions x.

Algorithm 1 Local search heuristic

Require: x^0, y^0, d^0, p^0, t^0, b^0, π^0, time$_{max}$, $n_{\min}$, $n_{\max}$, notImpr, tabuListSize
  $g := 0$, time $:= 0$, $n_{\text{fixed}} := n_{\min}$, notImpr $:= 0$, $z^* := -$INF, tabuList $:= \emptyset$
  repeat
    $p^g :=$ Price sampling(t^{g-1}, p^{g-1}, d^{g-1}) *[section 3.3.1]*
    $\{d^g, b^g\} :=$ Logit models(p^g) *[demand and recapture ratios for the sampled price based on equations (10) and (11)]*
    Ł := VNS - Fixing(x^{g-1}, t^{g-1}, d^{g-1}, n_{fixed}) *[selection of fixed assignments - section 3.3.2]*
    $\{x^g, y^g, \pi^g, t^g\}$:= solve $z_{\text{FAM}}(p^g, d^g, b^g, L)$ *[solve FAM with the sampled price, demand, recapture ratios and fixed assignments]*
    if ($\bar{x}^g \notin$ tabuList) **then**
      tabuList := tabuList $\bigcup x^g$ *[Tabu search, section 3.3.3]*
      $\{p^g, d^g, b^g, \pi^g, t^g\}$:= solve $z_{\text{RMM}}(x^g, y^g)$ *[solve RMM with fixed capacity]*
      if ($z_{\text{RMM}} \geq z^*$) *[if a better solution is obtained]* **then**
        Update z^*
        VNS - Intensification: $n_{\text{fixed}} := n_{\text{fixed}} + 1$ when $n_{\text{fixed}} < n_{\max}$
        notImpr := 0
      else if (notImpr == 3) *[if no improvement is obtained in the last 3 iterations]* **then**
        VNS - Diversification: $n_{\text{fixed}} := n_{\text{fixed}} - 1$ when $n_{\text{fixed}} > n_{\min}$
        notImpr := notImpr $-$ 1
      end if
    end if
    $g := g + 1$
  until time $\geq$ time$_{max}$

4 The data instances

For this study, we used a dataset coming from a major European airline which was provided in the context of the ROADEF Challenge 2009[2]. The dataset provides information on the set of airports, flights, itineraries, and aircraft. The information on the itineraries include their forecasted demand and average prices for each class.

Using the dataset several data instances are generated to be used throughout the paper as provided in Table 3. The data instances are presented with the number of airports, flights, the average number of flights per route, the average demand per flight and the available fleet. The information on the fleet includes the number of types of aircraft and the seat capacity of each type of aircraft. For some data instances the demand is low and the available aircraft are of small size. There are also larger instances with higher demand and bigger aircraft. Instances 20-25 are distinguished from the first 19 since they are large instances which generate more complex problems.

5 Performance of the heuristic approaches

In this section we present results for the three heuristic approaches presented in section 3. All the models are implemented in AMPL[3]. BONMIN runs over the full integrated model presented in section 2. The sequential approach and the local search heuristic

[2] `http://challenge.roadef.org/2009/en`
[3] `http://www.ampl.com`

Table 3: The data instances for the experiments

no	airports	flights	flights per route	demand per flight	fleet composition	
1	3	10	1.67	51.90	2	50-37
2	3	11	2.75	83.10	2	117-50
3	3	12	2.00	113.80	2	164-100
4	3	12	2.00	113.80	6	164-146-128-124-107-100
5	3	26	4.33	56.10	3	100-50-37
6	3	19	3.17	96.70	3	164-117-72
7	3	19	3.17	96.70	5	124-107-100-85-72
8	3	12	3.00	193.40	3	293-195-164
9	3	33	8.25	71.90	3	117-70-37
10	3	32	5.33	100.50	3	164-117-85
11	3	32	5.33	100.50	5	128-124-107-100-85
12	2	11	5.50	173.70	3	293-164-127
13	4	39	4.88	64.50	4	117-85-50-37
14	4	23	3.83	86.10	4	117-85-70-50
15	4	19	3.17	101.40	4	134-117-100-85
16	4	19	3.17	101.40	5	128-124-107-100-85
17	4	15	1.88	58.10	5	117-85-70-50-37
18	4	14	2.33	87.60	5	134-117-85-70-50
19	4	13	2.60	100.10	5	164-134-117-100-85
20	3	33	8.25	71.90	4	85-70-50-35
21	3	46	7.67	86.85	5	128-124-107-100-85
22	7	48	2.29	101.94	4	124-107-100-85
23	3	61	15.25	69.15	4	117-85-50-37
24	8	77	2.08	67.84	4	117-85-50-37
25	8	97	3.46	90.84	5	164-117-100-85-50

work with the models FAM and RMM. FAM is a MILP and solved using the GUROBI[4] solver. The RMM is a non-convex NLP and therefore BONMIN solver is used as done for the integrated model. BONMIN again serves as an heuristic since it is not designed for non-convex problems. For all the RMMs solved in the sequential approach and in the local search heuristic, BONMIN reported a 0% duality gap.

In order to test the performances of the three approaches we use the set of instances provided in Table 3. The time-limit for the solution of the integrated model with BONMIN is chosen as 24 hours in order to obtain feasible solutions to this highly complex problem. Maximum computational time allowed for the sequential approach and the local search heuristic is 1 hour. Sequential approach consists of one solution of FAM and RMM each and therefore does not need an excessive computational time. For the local search heuristic we also preferred to have a 1 hour limit in order to show that the resulting method is a practical method which can be used by practitioners. For all the approaches we report the time when the best solution is found. We note that since the considered revenue models are non-convex for all the approaches, the presented results are the best solutions obtained in the time limit, we can not talk about optimality.

The comparative results of the three approaches are presented in Table 4. The analysis of the results enables us to distinguish the following three cases.

[4]`http://www.gurobi.com/`

Table 4: Performance of the heuristic approaches

	BONMIN Integrated model		**Sequential approach (SA)**			**Local search heuristic** *Average over 5 replications*			
	Profit	Time (sec) *max 86,400*	Profit	% deviation from BONMIN	Time (sec) max 3,600	Profit	%deviation from BONMIN	%improvement over SA	Time (sec) *max 3,600*
1	15,091	2	15,091	0.00%	1	15,091	0.00%	0.00%	1
2	37,335	22	35,372	-5.26%	1	37,335	0.00%	5.55%	13
3	50,149	62	50,149	0.00%	1	50,149	0.00%	0.00%	1
4	46,037	2,807	43,990	-4.45%	1	46,037	0.00%	4.65%	3
5	70,904	1,580	69,901	-1.41%	1	70,679	-0.32%	1.11%	6
6	82,311	1,351	82,311	0.00%	1	82,311	0.00%	0.00%	1
7	87,212	32,400	84,186	-3.47%	1	87,212	0.00%	3.59%	60
8	779,819	8,137	779,819	0.00%	1	779,819	0.00%	0.00%	1
9	135,656	666	135,656	0.00%	2	135,656	0.00%	0.00%	2
10	107,927	482	107,927	0.00%	1	107,927	0.00%	0.00%	1
11	85,820	31,705	85,535	-0.33%	2	85,820	0.00%	0.33%	88
12	858,544	5,598	854,902	-0.42%	1	858,544	0.00%	0.43%	1
13	112,881	32,713	109,906	-2.64%	1	112,881	0.00%	2.71%	151
14	85,808	10,643	82,440	-3.93%	1	85,808	0.00%	4.09%	9
15	49,448	33	49,448	0.00%	1	49,448	0.00%	0.00%	1
16	38,205	240	37,100	-2.89%	1	38,205	0.00%	2.98%	1
17	27,076	35	27,076	0.00%	1	27,076	0.00%	0.00%	1
18	45,070	78	44,339	-1.62%	1	45,070	0.00%	1.65%	1
19	26,486	13	26,486	0.00%	1	26,486	0.00%	0.00%	1
20	146,773	30 846	146,464	-0.21%	1	147,506	0.50%	0.71%	406
21	194,987	4,963	210,134	7.77%	10	214,251	9.88%	1.96%	1,499
22	152,126	68,864	158,978	4.50%	2	159,258	4.69%	0.18%	39
23	227,643	40,862	226,615	-0.45%	12	227,284	-0.16%	0.30%	1,283
24	153,384	59,708	154,301	0.60%	4	158,099	3.07%	2.46%	2,314
25	313,943	82,780	331,920	5.73%	13	332,744	5.99%	0.25%	1,451

5.1 Case 1 - Easy instances with no improvement due to the integrated model

For the first 19 test cases, BONMIN reports 0% duality gap for the integrated model. For 9 of these instances (1, 3, 6, 8, 9, 10, 15, 17, 19), the integrated model does not improve the solution of the sequential approach. In other words, these instances does not show the superiority of simultaenous decision making on pricing and schedule planning. These cases are signified by a gray row color in Table 4. Since the solution of the sequential approach is the same as the integrated model solution, the local search heuristic stops after one iteration. As mentioned earlier, the local search heuristic solves the sequential approach as the first iteration. The computational time needed is a few seconds for those instances. This implies an order of magnitude reduction for the instances 3, 15, 17, 19. The gain of computational time is even more evident for the instances 6, 8, 9, and 10 with 2 to 3 order of magnitude.

5.2 Case 2 - Easy instances with an improvement due to the integrated model

Among the easy instances, the integrated model results with a superior solution compared to the sequential approach for the instances 2, 4, 5, 7, 11, 12, 13, 14, 16, and 18. The computational time needed for the sequential approach is again a few seconds. However it cannot reach the quality provided by the integrated model. The deviation of the sequential approach from the best solution can be up to 5.26%. The local search heuristic is able to find the best solution provided by the integrated model in a significantly reduced computational time. This reduction can be up to 4 order of magnitude as for instance 14. This shows that the local search mechanisms are successful to improve the sequential approach solution in a reasonable computational time. There is only one instance, 5, where the solution of the local search heuristic deviates (0.32%) from the solution of the integrated model provided by BONMIN.

5.3 Case 3 - Complex instances

The last 6 instances are larger compared to the first 19. The generic solver BONMIN reports a duality gap for these instances when solving the integrated model. The sequential approach runs maximum quarter of a minute and provides better feasible solutions in 4 of these instances. The local search heuristic performs better compared to the sequential approach in all the instances. The highest improvement is for experiment 24 with 2.46%. Similarly it outperforms the solutions provided by BONMIN on the integrated model except the instance 23 where there is a deviation of 0.16%. The local search heuristic has a computational time less than 40 minutes. This implies a time reduction of 1 to 3 order of magnitude compared to BONMIN. All in all, the local search heuristic provides better feasible solutions compared to both of the approaches. It can be used for large size instances where available solvers cannot provide good quality solutions. Therefore the local search heuristic enables to understand the added value of the integrated modeling framework for realistic size problems and can be used in decision making.

5.4 Added value of the spill based local search

As mentioned in section 3, local search heuristic involves mechanisms which enable to visit neighborhood solutions in an intelligent way based on the spilled number of passengers. In order to quantify the advantage of applying these local search rules, the local search heuristic is tested against its counter part with a fully random local search. The prices of the itineraries are uniformly drawn between the lower and upper bounds. Similarly the fixing of assignments is done randomly regardless of the spill values.

Table 5: Improvement due to the neighborhood search based on spill

	Sequential approach (SA)	**Random neighborhood**		**Neighborhood based on spill**		**% Improvement over random neighborhood**	
	Profit	Profit	Time(sec)	Profit	Time(sec)	Quality of the solution	Reduction in time
2	35,372	37,335	116	37,335	13	-	89.10%
4	43,990	44,302	27	46,037	3	3.92%	89.47%
5	69,901	*SA*	3,600	70,679	6	1.11%	99.83%
7	84,186	85,335	1,649	87,212	60	2.20%	96.35%
11	85,535	*SA*	3,600	85,820	88	0.33%	97.54%
12	854,902	*SA*	3,600	858,545	1	0.43%	99.97%
13	109,906	110,868	2,617	112,881	151	1.82%	94.23%
14	82,440	84,938	2,073	85,808	9	1.02%	99.57%
16	37,100	38,205	6	38,205	1	-	80.65%
18	44,339	45,070	358	45,070	1	-	99.72%
20	146,464	*SA*	3,600	147,506	406	0.71%	88.72%
21	210,134	*SA*	3,600	214,251	1,499	1.96%	58.36%
22	158,978	*SA*	3,600	159,258	39	0.18%	98.91%
23	226,615	*SA*	3,600	227,284	1,283	0.30%	64.36%
24	154,301	154,373	2,572	158,099	2,314	2.41%	10.03%
25	331,920	*SA*	3,600	332,744	1,451	0.25%	59.69%

The comparative results between the random neighborhood and the one based on spill is presented in Table 5. The instances where the sequential approach and the integrated model result with the same solution are omitted since in this case the local search heuristic is equivalent to the sequential approach.

Both versions of the local search heuristic have a time limit of 1 hour and the presented results are the average values for 5 replications of each. For 8 of the 16 instances (white rows), the random neighborhood does not improve the initial solution which is the same as the sequential approach in 1 hour. The neighborhood based on spill provides a better quality solution compared to the random neighborhood in 13 of the instances. The maximum improvement obtained in the profit is 3.92% (instance 4) and on the avereage this improvement is around 1.3%. In all of the 16 instances the spill based local search reduces the computational time considerably. The reduction in time can be up to 3 orders of magnitude as for instances 5 and 12. Therefore, the information provided by the demand model on the spill guides the heuristic method in the right direction and generates better feasible solutions in less computational time.

6 Conclusions and future research

In this paper a local search heuristic method is presented for the solution of the integrated airline scheduling, fleeting and pricing model. The main motivation for the heuristic is to obtain good quality feasible solutions in a reasonable computational time for large instances. The iterative process is carried out over two simplified versions of the integrated model and local search mechanisms are employed to explore better feasible solutions. The local search mechanisms are based on the information provided by the demand model on spill. This is an important feature of the heuristic approach which explores better feasible solutions in less computational time compared to a fully random neighborhood search. The resulting heuristic is practical and provides insights about the added value of the integrated approach for large size realistic problems.

The performance of the local search heuristic is compared to an available MINLP solver BONMIN and a sequential approach that represents the current practice of airlines. The local search heuristic outperforms the sequential approach when there is a potential gain from the simultaneous decision making. Otherwise, if there is no potential, it is equivalent to sequential approach. For large size instances it outperforms both of the other approaches. It is able to find better feasible solutions in a reasonable computational time.

The performance of the presented heuristic is evaluated in terms of the best feasible solution found. Since the problem is non-convex no evaluation could be done in terms of the duality gap. Therefore a potential future research is the extension of the study to obtain a valid upper bound through appropriate decomposition methods and/or transformations of the mathematical model. In the literature there are studies that come up with approximations to deal with the complexity of non-convex MINLPs. We refer to Nowak (2005) for a comprehensive set of methods for solving non-convex mixed integer nonlinear programs. Some studies present convex under estimation techniques for the non-convex functions in order to obtain valid bounds to the original problem (Gangadwala et al., 2006;Ballerstein et al., 2011). D'Ambrosio and Lodi (2011) present an overview on the available tools for convex and non-convex MINLPs. D'Ambrosio et al. (2012) develop an iterative technique for a non-convex MINLP based on a convex approximation of the model and a non-convex nonlinear program (NLP) that is obtained by fixing the integer part of the problem.

As mentioned previously, the non-convexity is due to the explicit logit model. Convexification techniques should be investigated to transform the logit model in to a convex/concave one.

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